APOLARITY, HESSIAN AND MACAULAY POLYNOMIALS

LORENZO DI BIAGIO AND ELISA POSTINGHEL

Abstract. A result by Macaulay states that an Artinian graded Gorenstein ring R of socle dimension one and socle degree δ can be realized as the apolar ring $\mathbb{C}[\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}]/f^{\perp}$ of a homogeneous polynomial f of degree δ in $x_0, \dots, x_n$. If R is the Jacobian ring of a smooth hypersurface $g(x_0, \dots, x_n) = 0$ then δ is equal to the degree of the Hessian polynomial of g. In this paper we investigate the relationship between f and the Hessian polynomial of g.

1. Introduction: the problem

Let $g \in R = \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial. The ideal $J(g)$ generated by the partial derivatives of g is called the Jacobian ideal, or the gradient ideal, of g. It is object of many studies, but so far it has not been completely understood. In the smooth case this ideal contains a power of the irrelevant ideal, so it has maximum depth in the coordinates ring and it is generated by a regular sequence. The associated ring $R(g) = R/J(g)$, the so called Jacobian ring of g, is an Artinian Gorenstein ring. Formal definitions can be found in Section 3.

A nice property is described in a classical theorem due to Macaulay (Theorem 2.2): there exists a homogeneous polynomial (the Macaulay polynomial) f such that $J(g)$ is equal to $f^{\perp}$, where $f^{\perp} \subset T = \mathbb{C}[\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}]$ (upon identyfing x_i with $\frac{\partial}{\partial x_i}$).
It is not so immediate to compute by hand the Macaulay polynomial associated to a given Artinian Gorenstein graded ring, but in the case of the Jacobian ring it seems natural to look at the Hessian polynomial $\mathrm{Hess}(g)$ of g since it has the right degree. It actually turns out that if g is a Fermat polynomial, then $\mathrm{Hess}(g)$ and the Macaulay polynomial associated to $R(g)$ coincide, up to scalars (see Example 3.1). Therefore we ask ourselves if $\mathrm{Hess}(g)$ is always the Macaulay polynomial (up to scalar multiplication) associated to g. A first naive conjecture is the following: $J(g) = \mathrm{Hess}(g)^{\perp}$, for every smooth homogeneous polynomial $g \in R$.

We will see in Section 3 that the question is not well-posed, anyway the answer is 'no' in general, but 'yes' for certain cases.

In Section 4 we will see how to use the computer algebra system CoCoA to attack this problem.

In Section 5 we will study the question for binary forms, giving a complete answer for forms of degree 3 and 4.

2000 *Mathematics Subject Classification.* Primary 14N15; Secondary 14J70.
Key words and phrases. Apolarity, Macaulay correspondence, Jacobian ring, Hessian polynomial.

In Section 6 we will completely answer the question in the planar cubic case.

In Section 7 we will present a first attempt to study planar quartics, giving some specific examples.

2. Preliminaries

Let k be an algebraically closed field of characteristic zero. Let $S := k[X_0, \dots, X_n]$ be the polynomial ring in $n+1$ variables, let $T := k[\partial_0, \dots, \partial_n]$ be the k-algebra generated by the partial derivatives ∂_i, where $\partial_i := \frac{\partial}{\partial X_i}$. S and T are naturally graded rings and we denote by S_d and T_d their degree d part, which is of course a k-vector space of dimension $\binom{n+d}{d}$.

By the natural differentiation action of T on S we can view S as a T-module. Since $\operatorname{char}(k) = 0$, this action is equivalent to the *contraction action* by [5, Def. 1.1]. Analogously we can think of S as the algebra of partial derivatives on T, hence we can also view T as an S-module. These two actions define a perfect pairing between homogeneous forms of degree j (see [3, sect. 2.2]):

$$S_j \times T_j \to k. \tag{1}$$

Given $f \in S$ and $g \in T$ we will say that g is *apolar* to f if $g \cdot f = 0$.

Remark 2.1. If f, g are homogeneous of the same degree then $f \cdot g = g \cdot f$.

Regarding S as a (left) T-module, let $I \subseteq T$ be an ideal and let $M, P \subseteq S$ be T-submodules of S. Recall that $(P :_I M) := \{i \in I | iM \subseteq P\}$ is an ideal of T contained in I. Analogously, if $M \subseteq P$, then recall that $(M :_P I) := \{p \in P | ip \in M \ \forall i \in I\}$ is a T-submodule of P.

In particular for any polynomial $f \in S \setminus \{0\}$ we will denote by $f^\perp$ the ideal of T of forms apolar to f, i.e. $f^\perp := \operatorname{Ann}(f) = \{g \in T | g \cdot f = 0\} = (0 :_T Tf)$. Let $T_f := T/f^\perp$; since $\sqrt{f^\perp} = (\partial_0, \dots, \partial_n)$ then T_f is an Artinian (0-dimensional) local ring.

Recall that a zero-dimensional local ring A is Gorenstein if and only if its *socle* (i.e. the annihilator of the unique maximal ideal) is simple (cf. [4, Prop. 21.5]). If moreover the ring is graded, we will call the *socle degree* the maximum integer j such that $A_j \neq 0$. Recall the following theorem (see [4, theorem 21.6], [6, §60ff], [5, lemma 2.12]):

Theorem 2.2 (Macaulay). *With notation as above, there is a one-to-one inclusion reversing correspondence between finitely generated nonzero T-submodules $M \subseteq S$ and ideals $I \subseteq T$ such that $I \subseteq (\partial_0, \dots, \partial_n)$ and T/I is a local, Artinian ring, given by*

$$M \mapsto (0 :_T M), \textit{the annihilator of } M \textit{ in } T;$$

$$I \mapsto (0 :_S I), \textit{the submodule of } S \textit{ annihilated by } I.$$

More precisely, ideals I as above such that T/I is local Artinian Gorenstein correspond to principal submodules Tf for some element $f \in S \setminus \{0\}$ (i.e. $I = f^\perp$).

Corollary 2.3 (Macaulay)**.** *The homogeneous ideals I as in Theorem 2.2 such that T/I is graded local Artinian Gorenstein and of socle degree j correspond to principal submodules Tf, where f is a homogeneous polynomial in S_j.*

Definition 2.4. We will call *Macaulay polynomial* associated to T/I the polynomial f associated to T/I (up to scalar multiplication) as in Corollary 2.3.

Remark 2.5. Some authors refer to the *Macaulay polynomial* as the *dual socle generator* of T/I.

Remark 2.6. If f is a homogeneous polynomial of degree j, then the Hilbert function $H(T_f)$ is symmetric with respect to $j/2$ (cf. [5, p. 9]). Hence by Corollary 2.3, if $I \subseteq (\partial_0, \dots, \partial_n)$ is a homogenous ideal of T and T/I is (graded) local Artinian Gorenstein and of socle degree j, then its Hilbert function $H(T/I)$ is symmetric with respect to $j/2$.

Given $I \subseteq (\partial_0, \dots, \partial_n)$ homogeneous ideal such that T/I is local Artinian Gorenstein and of socle dimension j, one can wonder if there is a simple way to determine the associated Macaulay polynomial $f \in S_j$. In fact the following holds:

Proposition 2.7. *The Macaulay polynomial associated to T/I is any nonzero element of $(0 :_{S_j} I_j)$.*

Proof. By Corollary 2.3, we know that $(0 :_S I)$ is a principal T-submodule generated by f, hence

$$(0 :_S I) = Tf = T_0 f \oplus \ldots \oplus T_j f = (0 :_{S_j} I) \oplus \cdots \oplus (0 :_{S_0} I),$$

and $T_0 f = kf$, the k-vector space of dimension 1 generated by f. Therefore any nonzero element of $T_0 f = (0 :_{S_j} I) = \{s \in S_j | is = 0 \ \forall i \in I\}$ can be chosen as f. Moreover since T/I has socle degree j and socle dimension 1, then I_j is a k-vector subspace of T_j of codimension 1, hence, by (1), $(0 :_{S_j} I_j)$ has dimension 1. Since $(0 :_{S_j} I_j) \supseteq (0 :_{S_j} I)$ and they have the same dimension we have that $(0 :_{S_j} I_j) = (0 :_{S_j} I)$. □

3. Jacobian ring and Hessian polynomial

Let $R := k[x_0, \dots, x_n]$ and let R_d its homogeneous degree d part. Let $g \in R_d$ defining a smooth hypersurface $V(g) \subset \mathbb{P}^n = \mathbf{Proj}(R)$. Let $J(g)$ be the *Jacobian ideal* of g, i.e. the homogeneous ideal in R generated by the partial derivatives $\frac{\partial g}{\partial x_0}, \dots, \frac{\partial g}{\partial x_n}$. Since g is smooth, then $\sqrt{J(g)} = (x_0, \dots, x_n)$, hence the *Jacobian ring* $R(g) := R/J(g)$ is graded, Artinian and local. It is well-known that in this case $R(g)$ is also Gorenstein and of socle degree $(n+1)(d-2)$.

Clearly R can be thought of as the k-algebra of partial derivatives on S, by the action $g(x_0, \dots, x_n) \cdot f(X_0, \dots, X_n) = g(\partial_0, \dots, \partial_n)(f(X_0, \dots, X_n))$ for every $g \in R, f \in S$. In this way we can identify R with T. Therefore, given $g \in R_d$, $g \neq 0$, the Jacobian ring $R(g)$ satisfies the hypotheses of Corollary 2.3, i.e. we can associate to $R(g)$ its Macaulay polynomial, namely the homogeneous polynomial $f \in S$ of degree $(n+1)(d-2)$ such that, under the natural identification of R with T, $J(g) = f^{\perp}$. We will call this f

the Macaulay polynomial associated to g, meaning that f is the Macaulay polynomial associated to $R(g)$. We will denote it by $f = \mathrm{Mac}(g)$.

Given $g(x_0, \dots, x_n)$ as before, we can also consider the Hessian polynomial of g, $\mathrm{Hess}(g) := (\widetilde{\mathrm{Hess}}(g))(X_0, \dots, X_n)$, where $\widetilde{\mathrm{Hess}}(g)$ is the determinant of the Hessian matrix, i.e. the matrix of partial, second order derivatives of $g(x_0, \dots, x_n)$. We will always consider $\mathrm{Hess}(g)$ up to scalar multiplication. Since g is nonsingular, by [1, §2.2] we have that $\mathrm{Hess}(g) \neq 0$. Moreover $\mathrm{Hess}(g)$ is homogeneous and $\deg(\mathrm{Hess}(g)) = (n+1)(d-2)$.

As we mentioned in the introduction, we first ask ourselves whether $\mathrm{Hess}(g)$ is the Macaulay polynomial (up to scalar multiplication) associated to g. Notice that checking if $\mathrm{Hess}(g)$ is the socle generator of the Jacobian ring associated to g is not a difficult task: by Proposition 2.7 and Remark 2.1 it is necessary and sufficient to check that $\mathrm{Hess}(g)$ kills all the forms in $J(g)_{(n+1)(d-2)}$.

Example 3.1. Let $g = x_0^d + \ldots + x_n^d \in R$ be the Fermat polynomial of degree d in $n+1$ variables. Then $J(g) = (x_0^{d-1}, \dots, x_n^{d-1})$ and $\mathrm{Hess}(g)$ is a monomial, $\mathrm{Hess}(g) = (d(d-1))^{n+1} X_0^{d-2} \cdot \dots \cdot X_n^{d-2}$. In this case $\mathrm{Hess}(g)$ is the Macaulay polynomial associated to g. In fact for any monomial $p \in S$ of degree $n(d-2)-1$, $\mathrm{Hess}(g) \cdot x_i^{d-1} p = 0 \Leftrightarrow \forall c \in k \setminus \{0\}, \mathrm{Hess}(g) \neq c X_i^{d-1} p$, and this inequality clearly always holds, for any c, p and i. Thus $R(g) = T/\mathrm{Hess}(g)^{\perp}$.

Example 3.2. Let $n = 2$, $g = (x_0 + x_1)^3 + x_1^3 + x_2^3$. Then $J(g) = (3(x_0 + x_1)^2, 3(x_0 + x_1)^2 + 3x_1^2, 3x_2^2)$ and $\mathrm{Hess}(g) = 216 X_0 X_1 X_2 + 216 X_1^2 X_2$. In this case $\mathrm{Hess}(g)$ is not the Macaulay polynomial associated to g. In fact, for example, $3(x_0 + x_1)^2 x_2 \in J(g)$, but $\mathrm{Hess}(g) \cdot 3(x_0 + x_1)^2 x_2 = 2592 \neq 0$. The Macaulay polynomial associated to g is (up to scalars) $X_0^2 X_2 - X_0 X_1 X_2$.

By Example 3.1 and Example 3.2, since the second one is simply obtained by performing a change of variables in the first one, it should be clear that the question whether $\mathrm{Hess}(g)$ is equal (up to scalars) to $\mathrm{Mac}(g)$ is not the right one.
Identifying R with T, as before, and taking $g \in R_d = T_d$, $g \neq 0$, first of all we should understand how $\mathrm{Hess}(g)$ and $\mathrm{Mac}(g)$ behave under a linear change of variables, i.e. under the action of $SL_{n+1}(k)$ on R_1. Let $\overline{X} = (X_0, \dots, X_n)$ and let $\overline{x} = (x_0, \dots, x_n)$. Let $A \in SL_{n+1}(k)$. The Hessian polynomial $\mathrm{Hess}(g)$ is covariant by the change of variables given by A, that is: $\mathrm{Hess}(g(A\overline{x})) = (\mathrm{Hess}(g(\overline{x}))(A\overline{X})$ (see [1, §2.1]). For the Macaulay polynomial the following lemma holds:

Lemma 3.3. *$Mac(g)$ is contravariant by the change of variables given by A, that is: $Mac(g(A\overline{x})) = Mac(g(\overline{x}))({}^t A^{-1} \overline{X})$.*

Proof. The basis $\overline{X} = (X_0, \dots, X_n)$ of S_1 and $\overline{x} = (x_0, \dots, x_n)$ of R_1 are dual to each other under the derivation actions (we are identifying R and T, as usual). Therefore the new basis $A\overline{x}$ of R_1 is dual to the basis ${}^t A^{-1} \overline{X}$ of S_1, hence for any polynomial $h \in S_d$ and $f \in R_d$ we have $h(\overline{X}) \cdot f(\overline{x}) = h({}^t A^{-1} \overline{X}) \cdot f(A\overline{x})$.

Moreover $J(f(A\overline{x}))$ is equal to $J(f)(A\overline{x})$, in fact $\nabla(f(A\overline{X})) = (\nabla f)(A\overline{X}) \cdot \nabla(A\overline{X}) = (\nabla f)(A\overline{X}) \cdot A$. Since A is invertible, the ideal generated by the

entries in $\nabla(f(A\overline{X}))$ is the same as the ideal generated by the entries in $(\nabla f)(A\overline{X}) \cdot A$. The thesis now follows. $\square$

Therefore, the new question we are interested in is "when are $\mathrm{Mac}(g)$ and $\mathrm{Hess}(g)$ projectively equivalent?".

Example 3.4. Going back to Example 3.2, as we have noted before, the polynomial g is obtained from the Fermat cubic polynomial $f := x_0^3+x_1^3+x_2^3$ by the linear change of variables given by the matrix

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $\mathrm{Hess}(g(\overline{x})) = \mathrm{Hess}(f)(A\overline{X})$, while $\mathrm{Mac}(g(\overline{x})) = \mathrm{Mac}(f)({}^tA^{-1}\overline{X})$. Therefore $\mathrm{Mac}(g)$ and $\mathrm{Hess}(g)$ are projectively equivalent by the linear change of variables given by ${}^tAA$, as it can be easily verified.

4. CoCoA

The freely available CoCoA system, implemented by an equipe in Genoa (see [2]), is well suited to perform calculations on polynomials. In particular it turned out to be very useful to experiment about the projective equivalence of the Hessian and the Macaulay polynomials associated to $g \in R_d$. This short program is an example on how we made calculations for Ex. 3.2:

`Use R::=QQ[x,y,z];` defines the ring in which we want to work;

`F:=(x+y)^3+y^3+z^3;` fixes the polynomial F;

`N:=Jacobian([F]); H:=Jacobian(N[1]); G:=Det(H);` returns the Hessian polynomial of F;

`J:=Ideal(N[1]);` computes the Jacobian ideal of F;

`Gort:=PerpIdealOfForm(G);` returns the ideal of derivations that kill G, i.e. $G^\perp$;

`Hilbert(R/J);` computes the Hilbert function of R/J;

`Hilbert(R/Gort);` computes the Hilbert function of $R/G^\perp$;

`InverseSystem(J,3);` returns the Macaulay polynomial associated to F (that is, the Macaulay polynomial associated to R/J, which has degree 3).

Also the function `DerivationAction(D,P);` is very useful: it returns the action of the derivation D on the polynomial P. For example, in Ex. 3.2 `DerivationAction(216xyz+216y^2z,3(x+y)^2z);` returns `2592`.

5. Binary forms of degree 3 and 4

In this section we will deal with homogeneous polynomials in two variables. For the sake of simplicity in this and following paragraphs we will not distinguish between upper-case and lower-case variables and moreover we will use the variables $x, y, z, \ldots$ instead of $x_0, x_1, x_2, \ldots$.

Let $f(x,y)$ be a nonsingular homogenous polynomial of degree d; $f(x,y) = g_1(x,y) \cdot \cdots \cdot g_d(x,y)$, where $g_i(x,y) = a_i x + b_i y$ are linear forms that are distinct up to constants.

If $d = 3$, since any set of three distinct points in $\mathbb{P}^1$ is projectively equivalent to any other set of this type, then every f as before is projectively equivalent to $x^3 + y^3$, the Fermat polynomial. Hence $\mathrm{Hess}(f)$ and $\mathrm{Mac}(f)$ are projectively equivalent for any f.

If $d = 4$, then any f, by the same token as before, is projectively equivalent, up to constants, to $f_a(x,y) = xy(x-y)(x+ay)$, where $a \in k, a \neq 0, -1$. Set $H_a := \mathrm{Hess}(f_a)$ and $M_a := \mathrm{Mac}(f_a)$. If $a = 1$ then f_1 is projectively equivalent to the Fermat polynomial $x^4 + y^4$. In this case we have

$$H_1 = -9(x^4 + 2x^2y^2 + y^4), \ M_1 = x^4 + 2x^2y^2 + y^4$$

and clearly these two polynomials are singular and equal (up to constants).

From now on we can suppose that $a \neq -2, -\frac{1}{2}, 1$, since the case $a = 1$ has just been analyzed, and when $a = -2, -\frac{1}{2}$ we have that f_a is projectively equivalent to $x^4 + y^4$. In general it can be easily seen that

$$H_a = -9x^4 - 12(a-1)x^3y - 6(2a^2 - a + 2)x^2y^2 + 12a(a-1)xy^3 - 9a^2y^4,$$

$$M_a = (a^2 + a + 1)x^4 - 2(a-1)x^3y + 6x^2y^2 + 2\frac{a-1}{a}xy^3 + \frac{a^2+a+1}{a^2}y^4.$$

Recall that if λ is the cross-ratio of four distinct points $V(f)$ in $\mathbb{P}^1$, then the associated j-invariant is defined as

$$j(f) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

The values of the j-invariant correspond to the projective equivalence classes of binary quartic forms. For example the class of the Fermat quartic have j-invariant equal to 1728.

Notice that H_a and M_a are two homogeneous polynomials of degree 4; hence they are projectively equivalent if and only if they have the same j-invariant. After computing the solutions $V(H_a)$ (and $V(M_a)$) of $H_a = 0$ (and $M_a = 0$, respectively) and the cross-ratios, we get that

$$j(H_a) = 2^8 \frac{(1 + a + a^2)^6}{a^2(2 + 5a - 5a^3 - 2a^4)^2},$$

$$j(M_a) = 2^8 \frac{27(1 + a + a^2)^3}{(2 + 3a - 3a^2 - 2a^3)^2}.$$

By our assumptions on a these numbers are well-defined. The equation $j(H_a) = j(M_a)$ gives us the twelve values of $a \in k \setminus \{0, 1, -1, -2, -\frac{1}{2}\}$ (counted with multiplicity) such that the corresponding binary forms have Hessian and Macaulay polynomials projectively equivalent. In particular for six of them we have that $j(f_a)$ is equal to 0, that H_a and M_a are nonsingular

and that $j(H_a) = j(M_a) = 0$; for the other six we have that $j(f_a) = 6912$, that H_a and M_a are nonsingular and that $j(H_a) = j(M_a) = 2304$.
Hence we have proved the following

Proposition 5.1. *There are only three projective equivalence classes of binary quartic forms such that the Macaulay and the Hessian polynomial are projectively equivalent.*

6. Plane cubics

In this section we will investigate the relations between the Hessian and the Macaulay polynomial associated to any smooth cubic in $\mathbb{P}^2$. It is the first interesting case to analyze in the plane, in fact in the conic case the two polynomials turn out to be trivially equal (up to scalar multiplication).

Recall that the *Hasse pencil* is the one-parameter family of curves defined by

$$f_a(x, y, z) = x^3 + y^3 + z^3 - 3axyz, \ a \in \mathbb{P}^1. \tag{2}$$

The starting point of our analysis is just to recall that any smooth cubic in $\mathbb{P}^2$ is projectively equivalent to a cubic in the Hasse pencil for a certain $a \in \mathbb{C}, a^3 \neq 1, a \neq \infty$. Hence we can reduce the problem of studying planar cubics to the analysis of the cubics in (2). If $a = 0$ we simply get the Fermat cubic and, as already seen in Example 3.1, $H_0 = M_0 = xyz$. From now on we will suppose $a \neq 0$. The Jacobian ideal of $f_a(x, y, z)$ is

$$J(f_a) = \left(x^2 - ayz, y^2 - axz, z^2 - axy\right).$$

The Hessian polynomial is

$$H_a := \mathrm{Hess}(f_a) = \begin{vmatrix} 2x & -az & -ay \\ -az & 2y & -ax \\ -ay & -ax & 2z \end{vmatrix} = (8 - 2a^3)xyz - 2a^2(x^3 + y^3 + z^3).$$

Notice that H_a is equal (up to scalar multiplication) to $f_b(x, y, z)$, with $b = (4 - a^3)/3a^2$.
By proposition 2.7 we look for the Macaulay polynomial M_a of f_a among all cubic forms apolar to $J(f_a)_3$. We notice that M_a has to be symmetric with respect to x, y, z since the generators of $J(f_a)_3$ are. Therefore, with a simple computation, it is straightforward to see that M_a is equal (up to scalar multiplication) to f_c, with $c = -2/a$.

If $a \neq 0$, once we know the Hessian and the Macaulay polynomials associated to a nonsingular Hasse cubic $f_a = 0$, we can determine if they coincide, or if they are projectively equivalent. The solutions of the equation

$$\frac{4 - a^3}{3a^2} = -\frac{2}{a}$$

correspond to the unique cubics of the Hasse pencil having $H_a = M_a$ and $a \neq 0$: they are the ones corresponding to the values

$$a = -2, a = 1 - \sqrt{3}, a = 1 + \sqrt{3}.$$

We know that two planar cubics have the same j-invariant if and only if they are projectively equivalent (or isomorphic). Hence we can study the behavior of the Macaulay and the Hessian polynomials for a representative

in each isomorphism class. In any case the equation (2) allows us to make explicit computations. There is a formula for the j-invariant of a cubic of the form (2) (see [8], Lemma 2.2):

$$j(f_a) = -\frac{a^3(a^3+8)^3}{(1-a^3)^3}.$$

Therefore, imposing $j(H_a) = j(M_a)$, we get a finite number of solutions. This gives us explicitly the cubics in the Hasse pencil having the Hessian and Macaulay polynomials projectively equivalent. In this case, there are only three values for the j invariant such that the corresponding cubics verify the property: $j = 64, j = \omega, j = -\omega$, for a certain $\omega \in \mathbb{C}$.
In all the remaining cases, that are infinitely many, H_a and M_a are not projectively equivalent.
We have proved the following

Proposition 6.1. *There are only four projective equivalence classes of plane cubics such that the Macaulay and the Hessian polynomial are projectively equivalent.*

We close this section going back to the Fermat cubic case, making some interesting remarks.

Definition 6.2. An *equianharmonic* cubic is a planar smooth cubic C verifying one of the following equivalent conditions:

(1) $j(C) = 0$, i.e. it is projectively equivalent to the Fermat cubic;
(2) the polynomial defining C can be written as a sum of three powers of independent linear forms;
(3) the coefficients of the polynomial defining C verify the vanishing of S_4, the *quartic invariant of ternary cubics* ([9]). Furthermore if S_6 is the sextic invariant of ternary cubics, then one has $j = S_4^3/S_6^2$;
(4) in the space of cubics $\mathbb{P}^9$, C sits on the variety $\mathrm{Sec}_2(V_3)$ of trisecant planes to the 3-ple Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$, which is a quartic hypersurface of equation $S_4 = 0$.

An interesting question is the following: how many cubics of the Hasse pencil are equianharmonic? Notice that (2) is a line in the space of cubics $\mathbb{P}^9$ and hence we expect that it intersects $\mathrm{Sec}_2(V_3)$ in four points.
In fact, imposing $j(f_a) = 0$ we get four different solutions, corresponding to

$$a = 0, a = -2, a = 1 + \sqrt{3}i, a = 1 - \sqrt{3}i.$$

Notice that we already met the cases $a = 0, -2$: $H_0 = M_0$, $H_{-2} = M_{-2}$. For the other two values of a notice that $f_{1+\sqrt{3}i}$ and $f_{1-\sqrt{3}i}$ exchange their Hessian and Macaulay polynomial, that is: $H_{1+\sqrt{3}i} = M_{1-\sqrt{3}i}$ and $H_{1-\sqrt{3}i} = M_{1+\sqrt{3}i}$.

7. Plane quartics

In this section we will make some remarks about the relationship between the Hessian and the Macaulay polynomials for smooth quartics in $\mathbb{P}^2$. Since a general quartic can be written as a sum of at most six powers of linear forms (see, for example, the introduction in [7]), and - as already seen - we

can just analyze one quartic for each class of projective equivalence, we have the following cases:

(1) Fermat quartic: $x^4 + y^4 + z^4$. The Hessian polynomial coincides, up to a multiplicative constant, with the Macaulay polynomial: $x^2y^2z^2$ (this is a special case of Ex. 3.1).
(2) Caporali quartics: $x^4 + y^4 + z^4 + l(x, y, z)^4$, where $l(x, y, z) = ax + by + cz$ is a linear form, $a, b, c \in k$.
In this case, depending on a, b, c there are many examples for which the Hilbert functions of the apolar ring to the Hessian polynomial and to the Macaulay polynomial, respectively, are different. When this happens clearly the Hessian polynomial and the Macaulay polynomial cannot be projectively equivalent. Let us provide some explicit computations:
    - Let $a = b = c = 1$. Then the Hilbert function of the Jacobian ring (i.e. the apolar ring to the Macaulay polynomial by Theorem 2.2) is the following: $H(0) = 1, H(1) = 3, H(2) = 6, H(3) = 7, H(4) = 6, H(5) = 3, H(6) = 1, H(t) = 0$, for $t \geq 7$, while in the Hessian case there is just one difference: $H(3) = 10$.
    - Let $a = 1, b = 2, c = 0$. In this case the two Hilbert functions coincide.
(3) Clebsh quartics: $x^4 + y^4 + z^4 + l_1(x, y, z)^4 + l_2(x, y, z)^4$, where l_1, l_2 are linear forms. As before there are examples for which the Hilbert functions are the same and other examples for which they are different.

Another well-known famous quartic in the plane is the Klein quartic: $x^3y + y^3z + z^3x$. For this quartic the Hessian polynomial coincides, up to a multiplicative constant, with the Macaulay polynomial: $xy^5 + x^5z - 5x^2y^2z^2 + yz^5$. Notice that the same is true also for the Klein cubic in $\mathbb{P}^4$: $x^2y + y^2z + z^2w + w^2t + t^2x$, where its Hessian is $32x^3z^2 - 32xyz^3 + 32y^3w^2 + 32x^2w^3 - 32yzw^3 - 32xy^3t - 32x^3wt + 96xyzwt + 32z^3t^2 + 32y^2t^3 - 32zwt^3$.

8. Acknowledgements

Both the authors wish to warmly thank Prof. Edoardo Sernesi for having suggested this problem and for many helpful and fruitful discussions.

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Dipartimento di Matematica, “ Sapienza” - Università di Roma - Piazzale Aldo Moro 5, 00185, Roma, Italy
E-mail address, 1: `dibiagio@mat.uniroma1.it`
E-mail address, 2: `lorenzo.dibiagio@gmail.com`

Dipartimento di Matematica, Università “Roma Tre” - Largo S. Leonardo Murialdo 1, 00146, Roma, Italy
E-mail address, 1: `postingh@mat.uniroma3.it`
E-mail address, 2: `elisa.postinghel@gmail.com`